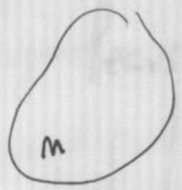
Jean Lannes, École Polytechnique Aug 26, 2009 (1)

"The Kervaire invariant and manifolds with boundary"

1) Recalls on Kervaire quadratic form and Kervaire invariant

(M, t, n)



$\dim M = 2n \qquad n > 0$

$x \in H^n M = H^n(M; \mathbb{Z}/2)^r$

$S^{2n} \to Th(M) \to Th(M) \wedge M_+ \to Th(M) \wedge K_n$

$(M, t, x) \longmapsto [M, t, x] \in \pi_{2n}(S \wedge K_n)$

$= F^1 \pi_{2n}(S \wedge K_n)$

$X \quad \pi_* X = F^0 \pi_* X \supset F^1 \pi_* X \supset \dots$

$K_n = K(\mathbb{Z}/2, n)$

$(\mathrm{Tor}_2(\mathbb{F}_2, \bar{H}^* K_n))^{2n+1}$

$\cup$

$Sq^{n+1} \otimes_A 2n$ $\qquad Sq^{n+1} \otimes_A 2n$

$(M, t, x) \qquad [M, t, x] \longmapsto \mathbb{Z}/2 \ni q_t(x) = \langle x \cup v_n, [M] \rangle$

$H^n M \xrightarrow{q_t} \mathbb{Z}/2$

$q_t(x+y) = q_t(x) + q_t(y) + x \cdot y$

q_t n-d quad for on $H^n M$

$(M, t, x$

$x \in H^n M$

$x = y|_M \qquad y \in H^n N$

$q_t(x) = 0$

$[M, t, x] = 0$ (2)

$H^n N \xrightarrow{i^*} H^n M$

$i : M \to N$

Then $I = \mathrm{im}(i^*)$; $I = I^\perp$ $i : M \to N$

↳ I is a lagrangian

$H^*(\ ; \mathbb{R})$

$q_t(I) = 0$

$(M, t) \quad H^n M, q_t$

if $(M, t) = \partial(N, T)$ then

$(H^n N, q_t)$ is neutral (split)

E K.v.r. $\qquad E \xrightarrow{q} K$ n-d quad. form

$q(x+y) = q(x) + q(y) + xy$

$(x, y) \longmapsto x \cdot y$

$E \times E \to K$ n.d sym. bil. form

$\exists\, I \subset E$ s.t. $I = I^\perp$, $q(I) = 0$

Rmk $\quad E \simeq H(I) \qquad I \oplus I^* \xrightarrow{q} K$

(x, ξ)

$q(x, \xi) = \langle x, \xi \rangle$

$0 \to I \to E \ (\simeq E^*) \to I^* \to 0$

$I = I^\perp$ means this seq. is exact

Claim it is possible to choose s

s.t. $q(s(I^*)) = 0$

$V \xrightarrow{q} K$ $\exists\ b: V \times V \to K$

$q(x) = b(x,x)$

Quadratic Witt group $WQ(K)$

= (monoid of iso. class of n d quad

fms for the orthogonal sum) / (metabol ones)

Kervaire invariants: $\Omega_{2n}^{fr} \to WQ(\mathbb{F}_2)$

$(M,t) \to (H^n M, q_t)$

E $\mathbb{F}_2$-v-s $E \xrightarrow{q} \mathbb{F}_2$ n-d quad. fm.

$q(x+y) = q(x) + q(y) + x.y$

$\exists\ I \subset E$ s.t. $I = I^\perp$ (bilinear lagrangian)

$e \in E$ $e \neq 0$, $\exists\ f \in E$ $e.f = 1$

$e.e = 0$ $\begin{array}{c|c} & e \quad f \\ e & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ f & \end{array}$ $q(\lambda x) = \lambda^2 q(x)$

$q(0) = 0$ e, f ℓ^t = ind t

$E = \langle e, f\rangle \oplus \langle e, f\rangle^\perp$

$q|_I : I \to \mathbb{Z}/2$ $\gamma: E \to \mathbb{F}_2$

$x, y \leftarrow \mathbb{F}_2$

$q(x+y) = q(x) + q(y)$

extension of $q|_I$ to E.

$\gamma(x) = u.x$ $u \in E$

$\exists\ u \in E$ s.t. $q(x) = u.x\ \forall\ x \in I$

1) $q|_I = 0$, $q(I) = 0$ $E \simeq H(I)$

2) $q|_I \neq 0$, $J \subset I$, $J := \operatorname{Ker} q|_I$

$\exists\ u' \in I$ s.t. $q(u') = 1$

$q(u') = 1$ $u.u' = 1$

$E = \langle u, u'\rangle \oplus \underbrace{\langle u, u'\rangle^\perp}_{J}$ $\cong H(J)$

$\begin{array}{c|c} & u \quad u' \\ u & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ u' & \end{array}$

sym. lin. form

$q(u+u') = q(u) + q(u') + u.u'$

$\mathbb{F}_2^2 \xrightarrow{q} \mathbb{F}_2$

$q(u') = 1$

$\begin{cases} q(u) = 0 \\ q(u) = 1 \end{cases}$

$q(x) = 1$ if $x \neq 0$

$q(0) = 0$

$E \cong H(\mathbb{F}_2) \oplus \underbrace{H(J)}_{\text{permanent}} = H(\mathbb{F}_2 \oplus J)$

dim $E = 2n$

$E \cong (n-1)\underset{\substack{\| \\ H(\mathbb{F}_2)}}{E_0} \oplus \begin{cases} E_0 \\ E_1 \end{cases}$

$E_1 \oplus E_1 \simeq E_0 \oplus E_0$

$WQ(\mathbb{F}_2) \simeq \mathbb{Z}/2 \quad [E_1]$

$\xrightarrow{\sim}$

$GWQ(\mathbb{F}_2) \simeq \mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z}/2$

Ω^K_{2n} [Kervaire equivalence]

K any commutative ring.

I a proj. K-module of even rank.

$P \xrightarrow{q} K$

$\Delta(P) = Z(C^{av}(P, q))$

<u>Claim:</u> $K \subset \Delta(P)$

$\Delta(P)$ is projective of rk 2

$K \underset{\text{étale}}{\hookrightarrow} \Delta(P)$ | $\Delta(I) \in H^1((\mathrm{Spec}\, K)_{et}; \mathbb{Z}_2)$